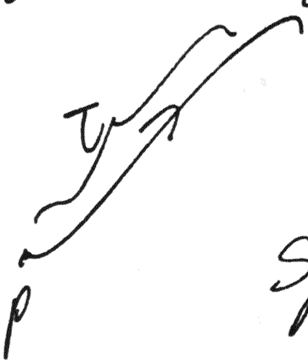
Chapter 2: Matrices & Strings

2.1 Path integrals of strings

. QFT: a theory of "particles"

If particle interactions are weak, we can consider the first quantized approach



$$X^\mu(\tau) \qquad \mu = 0, 1, \cdots, d-1$$

$$S_{particle} = -m \int \underbrace{dl}_{\text{proper length}}$$

$$= -m \int d\tau \, \frac{dl}{d\tau} = -m \int d\tau \sqrt{-\eta_{\mu\nu} \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau}} \quad (*)$$

1) Lorentz invariant
2) Correct equation of motion
3) Correct non-relativistic limit
4) Reparameterization invariant in τ

$$\tau \to \tau'(\tau)$$

$$X^\mu \to X^{\mu'} \quad \text{s.t.} \quad X^\mu(\tau) = X^{\mu'}(\tau')$$

For a quantum particle:

$$G(X,X') = \sum_{\substack{\text{path} \\ \text{from } X \text{ to } X'}} e^{iS_{\text{particle}}}$$

but "$\sqrt{\ }$" is awkward to deal with

rewrite S_{particle} as

$$S = \frac{1}{2}\int d\tau \left(e^{-1}(\tau)\,\eta_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau} - e(\tau)\,m^2\right)$$

Upon eliminating e, we go back to $(*)$

$$\Rightarrow G(X,X') = \int_{X_i^\mu = X'}^{X_f^\mu = X} DX^\mu\, De\; e^{iS_{\text{particle}}}$$

$\Rightarrow\ =$ Feynman propogator for a scalar field of mass m

Note:

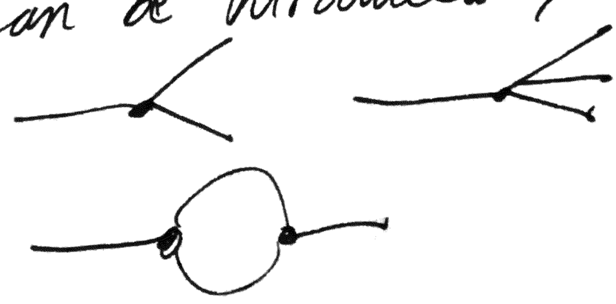
1) $e(\tau)$ is an intrinsic "vielbein" along the worldline

$\underline{\underline{h_{\tau\tau}}} = -e^2(\tau)$ ← intrinsic metric on the worldline

2) For curved spacetime, simply do

$$\eta_{\mu\nu} \to g_{\mu\nu}(X(\tau))$$

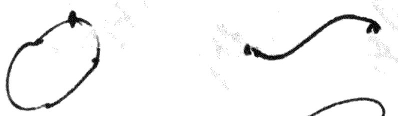
3) Interactions can be introduced by including vertices



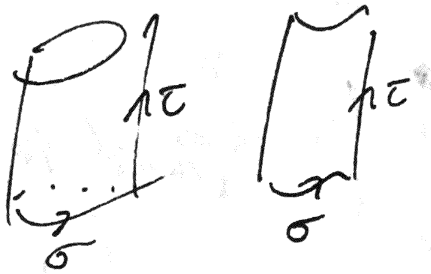
4) No general principle to restrict allowed types of vertices $\Leftrightarrow$ interactions in a QFT have to be specified by hand

- Strings:

1-d objects



world sheet



$\Sigma : X^\mu(\xi^a)$ $\qquad a = 0,1$

$\qquad\qquad\qquad \xi^a = (\sigma, \tau)$

(*) can be immediately generalized

$$S_{NG} = -T \int_\Sigma dA$$

$$= -T \int_\Sigma d^2\xi \sqrt{-\det\Big(\underbrace{\eta_{\mu\nu} \frac{dX^\mu}{d\xi^a} \frac{dX^\nu}{d\xi^b}}_{h_{ab}}\Big)}$$

h_{ab} induced metric

$$= -T \int_\Sigma d^2\xi \sqrt{-h}$$

Alternative form:

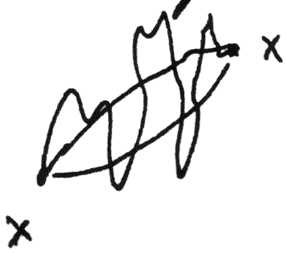
Introduce new auxiliary metric γ^{ab} "intrinsic metric on Σ"

$$S_p = -\frac{T}{2}\int_\Sigma d^2\xi \sqrt{-\gamma}\,\gamma^{ab} h_{ab}$$

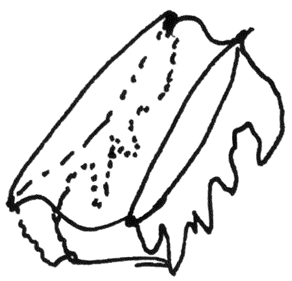
Quantum dynamics of string:

Integrate over all possible string trajectories

$\Leftrightarrow$ Integrate over all 2-d surfaces (weighted by e^{iS_p})



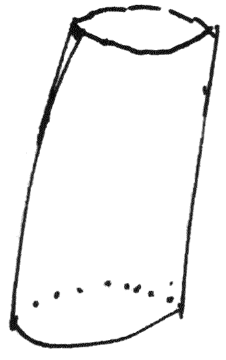
$\Rightarrow$

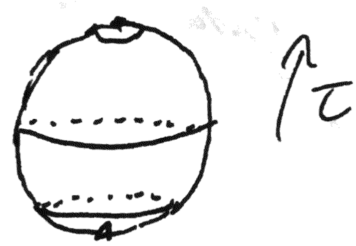


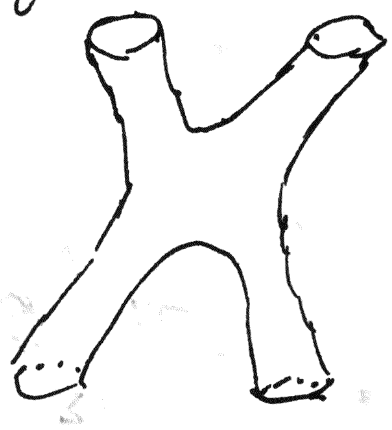
(hard to draw)

$$A = \int DX(\xi^a)\, D\gamma_{ab}\, e^{iS_p}$$

Some examples:





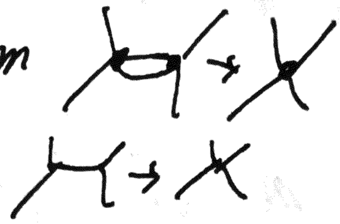


Remarks:

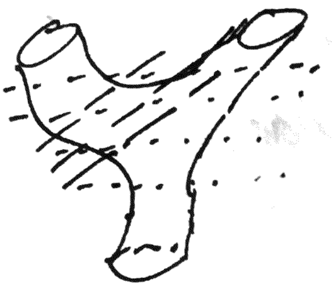
(a) Each such string diagram should be understood as representing integration of all continuous deformations of the corresp. surface, i.e. view each diagram as a rubber sheet (integrating over γ_{ab}, X^μ corresponds to arbitrary stretching)

b) Stretching a 2-d surface is much richer than stretching a line, leading to many important new features of string theory

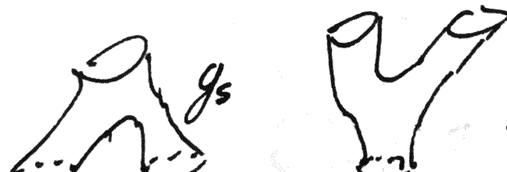
(1) No sharp interacting vertices

, UV divergences come from 

vs

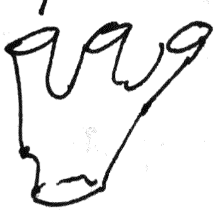
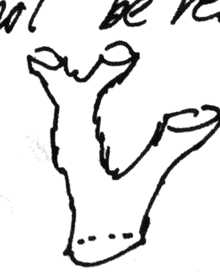
 $\Rightarrow$ string theory has excellent UV behavior (in fact UV finite)

(2) Interactions of strings are essentially unique: All 2-D surfaces can be built up from:

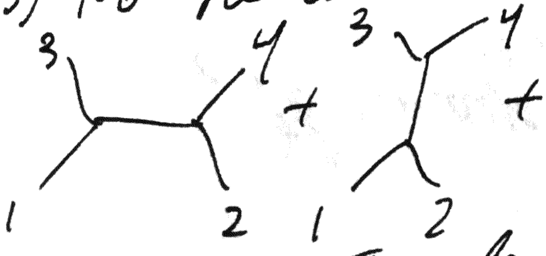


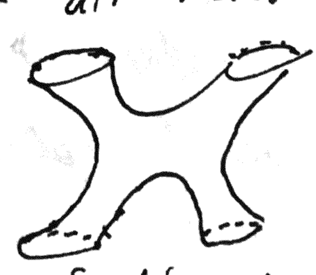
$\underline{g_s}$ ← key fundamental constant in string theory

For particles,  cannot be reduced to 

but for strings  → 

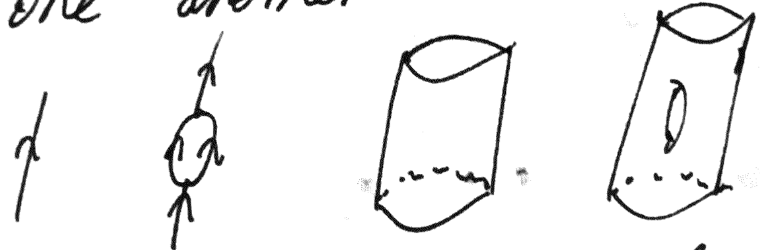
(3) No need to sum over different channels



$+ \cdots$ vs 

→ Tree level scattering of strings has a single diagram

c) Diagrams of different loops are given by ~~seamless~~ surfaces of different topologies, which cannot be continuously deformed to one another



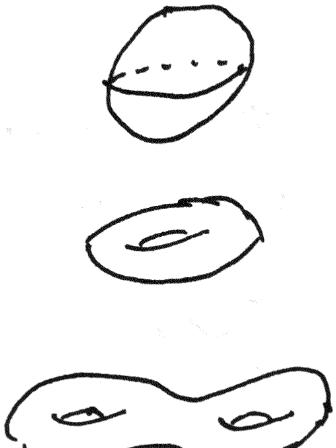
d) Basic theorem of topology:

Orientable closed 2-d surfaces are classified by a single integer h called genus, which is the number of holes (handles)

genus 0:

1:

2:



genus = # of loops

sum over loops = sum over topologies

$\Rightarrow$ at each loop order we have a single diagram

• g_s -dependence $\Rightarrow$ at each loop multiply by g_s^2 $\Rightarrow$ for h-loops g_s^{2h}

tree-level amplitude for n strings: g_s^{n-2}

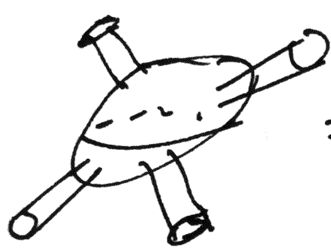
$$A_n = g_s^{n-2}\left(A_n^{(0)} + g_s^2 A_n^{(1)} + \cdots + g^{2h} A_n^{(h)} + \cdots\right)$$
$$= \sum_{n=0}^{\infty} g_s^{n-2+2h} A_n^{(h)}$$

Path integrals over surfaces of genus h with n external strings

also applies to $n=0,1,2$ from unitarity

$n=0$  $\sim g_s^{-2}$

$n=1$  $\sim g_s^{-1}$ $\simeq$  (i)

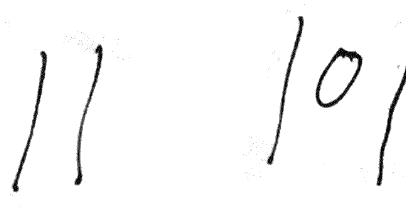
$A_n^{(0)} =$  $\Rightarrow$ Each external leg can be ~~considered~~ associated with g_s

Note: $2-n-2h = \chi_{n,h}$: Euler character for a surface of h handles and n boundaries

$$\Rightarrow A_n = \sum_{h=0}^{\infty} g_s^{-\chi_{n,h}} A_n^{(h)}$$

Open strings are the same story:

 g_o  g_o

 $\Rightarrow$ adding a loop $\Leftrightarrow$ adding a boundary

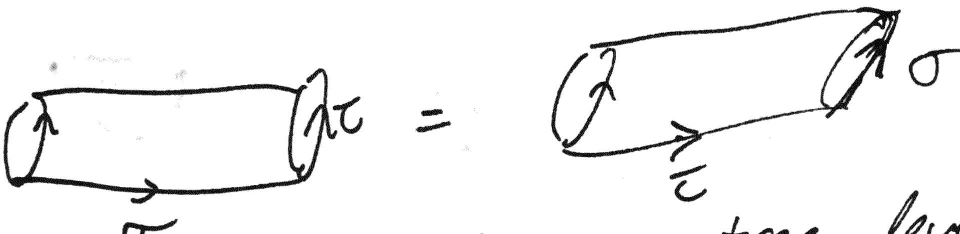
$\propto \times g_o^2$

$$A_n^{(open)} = \sum_{L=0}^{\infty} g_o^{\,n-2+2L} A_n^{(L)}$$

$n=0$:  g_o^{-2} (ii)
$L=0$

Now a profound statement:

A theory of open strings must contain closed strings



$\Rightarrow$ 1-loop open string = tree level propagation of closed string

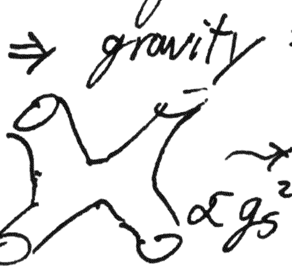
For a theory with <u>both</u> open and closed strings, comparing diagrams (i) and (ii) gives $g_s \propto g_o^2$.

Altogether: A_n contributions go as $g_s^{\,n_c + \frac{n_o}{2} - 2 + 2h + b}$

n_c: # closed strings, n_o: # open strings, h: # handles, b: # boundaries

Closed string excitations: graviton, ...

$\Rightarrow$ gravity $\Rightarrow G_N \sim g_s^2$

 $\propto g_s^2$ $\rightarrow$  $\propto G_N$

Open string excitations: A_μ gauge field

$g_{YM}^2 \propto g_o^2$

2.2 Matrix integrals in the large-N limit

A non-abelian gauge theory with $SU(N)$ gauge group

$$\mathcal{L} = -\frac{1}{4}\frac{1}{g_{YM}^2} \operatorname{Tr} F_{\mu\nu} F^{\mu\nu}$$

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]} - i[A_\mu, A_\nu]$$

$$A_\mu = (A_\mu)^a{}_b \qquad a,b = 1, \dots, N$$

$N \times N$ traceless hermitian matrices

$N=3$: gluon sector of QCD

't Hooft 1974: $N \to \infty$
expand in $1/N$

consider first a matrix scaler theory:

$$\mathcal{L} = -\frac{1}{g^2} \operatorname{Tr}\left[\frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi + \frac{m^2}{2}\Phi^2 + \frac{1}{4}\Phi^4\right]$$

$\Phi = \Phi^a{}_b$ $N \times N$ hermitian

$$(\Phi^a{}_b)^* = \Phi^b{}_a$$

ie.
$$\mathcal{L} = -\frac{1}{g^2}\left[\frac{1}{2}\partial_\mu \Phi^a{}_b \partial^\mu \Phi^b{}_a + \frac{1}{2}m^2 \Phi^a{}_b \Phi^b{}_a + \frac{1}{4}\Phi^a{}_b \Phi^b{}_c \Phi^c{}_d \Phi^d{}_a\right]$$

for <u>any</u> spacetime dimension d
$d=0$: Matrix integral
$d=1$: QM (Matrix)

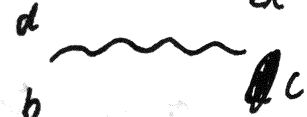
$\mathcal{L}$ is invariant under a $U(N)$ global symmetry

$$\Phi(x) \to U\Phi(x)U^\dagger$$

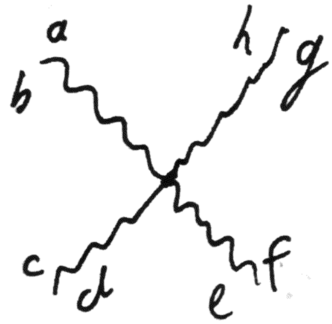
$$U \in U(N) \quad \text{const.}$$

Feynman rules:

$$\langle \Phi^a_b(x)\, \Phi^c_d(y) \rangle$$

$=$ 

$$= g^2\, \delta^a_d\, \delta^c_b\, G(x-y)$$ ← scalar propagator

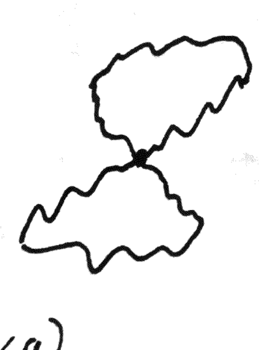


$$= \frac{1}{g^2}\, \delta^a_h\, \delta^c_b\, \delta^e_d\, \delta^g_f$$

vacuum process:

$$Z = \int D\Phi(x)\, e^{i\int d^dx\, \mathcal{L}}$$

$\log Z$ = sum of all connected diagrams with no external legs

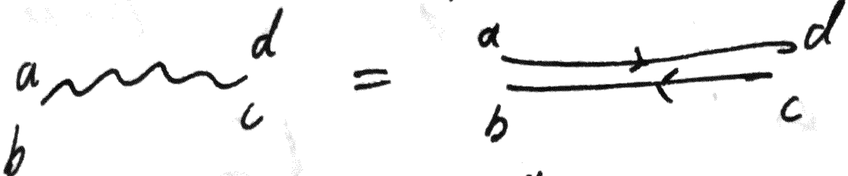
$=$  $+$ 

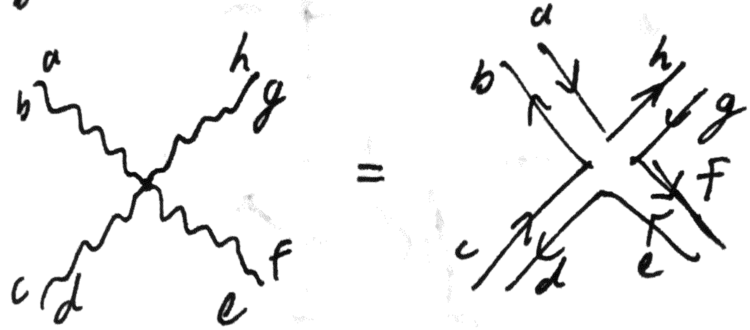
(a) $\qquad$ (b)

(a) $\propto g^2 N^3$ (planar)

(b) $\propto g^2 N$ (non-planar)

Trick (introduced by 't Hooft): double line notation

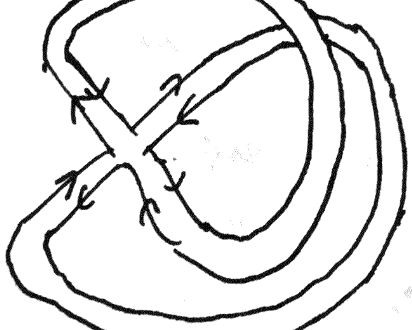




(i) a single line: index line
indices connected by a single line are contracted

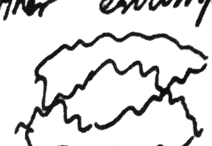
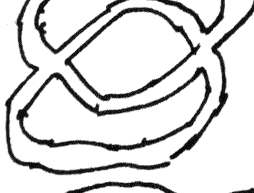
(ii) direction: from upper to lower indices

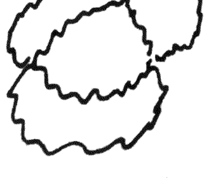
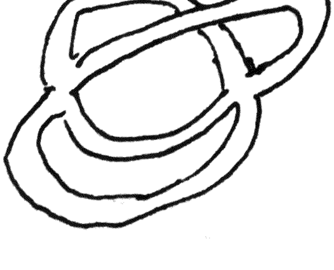
$3 \rightsquigarrow N^3$


$1 \rightsquigarrow N$

Each index loop gives $\sum_a \delta^a_a = N$

another example

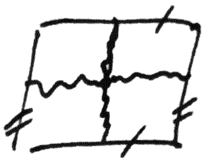
(c)  $\rightarrow$  $\rightsquigarrow N^4$

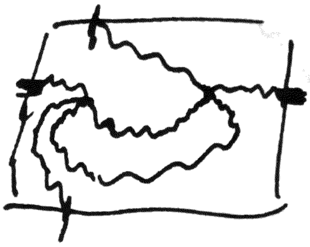
(d)  $\rightarrow$  $\rightsquigarrow N^2$

Empirical evidence:
non-planar diagrams
have smaller
N-dependence

Hints:

(1') These non-planar diagrams can be drawn on the torus without crossing lines

e.g. (b) can be drawn as: 

(d) can be drawn as: 

(2') The power of N for each diagram is equal to the number of faces after we straighten it.

Face: A connected subregion bounded by propagators in a diagram

For planar diagrams, the outside region also counts as a face.

double line: each face is bounded by an index loop

recall: an orientable closed 2-d surface is classified by its number of handles (genus) h

a genus-h surface $\Leftrightarrow$ polygon of $4h$ sides with opposite pairs identified

Generalize 1' and 2' to:

1. For any non-planar diagram, $\exists h$ s.t. the diagram can be drawn on a genus h surface but not lower.
   $\Rightarrow$ diagrams can be classified by genus h
2. N-dependence is given by # of faces on genus h surface

For a general diagram:

$$A \propto \underbrace{(g^2)^E}_{\substack{\#\text{ of}\\ \text{propogators}}} \underbrace{(g^{-2})^V}_{\substack{\#\text{ of interaction}\\ \text{vertices}}} \underbrace{(N)^F}_{\substack{\#\text{ of}\\ \text{faces}}} \qquad (*)$$

→ note this is unbounded

⇒ (naively) there is no sensible large N limit

However, note that the Feynman diagrams triangulate the surface. For any triangulation, the Euler characteristic is an invariant of the surface:

$$\chi = F+V-E = 2-2h$$

$$\Rightarrow A \propto (g^2)^{E-V} N^{F+V-E} E^{E-V}$$

$$= (g^2 N)^{E-V} N^{2-2h}$$

Now let $g^2 \to 0$, $N \to \infty$

but $\lambda \equiv g^2 N$ finite

$$\underbrace{E-V}_{\substack{\text{number}\\ \text{of indep.}\\ \text{momenta}}} = \underbrace{L}_{\#\text{ of loops}} - 1 \Rightarrow A \propto \lambda^{L-1} N^{2-2h}$$

sum over genus-h diagrams

$$\Rightarrow (\text{'t Hooft limit}) \quad \log Z = \sum_{h=0}^{\infty} N^{2-2h} f_h(\lambda)$$

$$f_h(\lambda) = \sum_{L=1}^{\infty} f_{hL} \lambda^{L-1}$$

In the $N \to \infty$ limit, Planar diagrams dominate

<u>N.B.</u> Though for general diagrams, the number of contributing diagrams increase <u>factorially</u> in N, for <u>planar</u> diagrams, this instead grows <u>polynomially</u>

We see $\log Z \propto N^2$, and from the Lagrangian we can see this too:

$$\mathcal{L} = -\frac{1}{g^2}\mathrm{Tr}\left(\frac{1}{2}(\partial\phi)^2 + \dots\right) = -\frac{N}{\lambda}\mathrm{Tr}(\cdots) \sim \mathcal{O}(N^2)$$

2) This discussion only depends on structure, not on detailed ~~[illegible]~~ form of $\mathcal{L}$ or its fields (spinor, vector, etc..)

$$\mathcal{L} = N\,\mathrm{Tr}(\cdots)$$

Last time:

$\mathcal{L} = N \, \mathrm{Tr}(\cdots)$ $\quad U(N)$ symmetry

$$\Rightarrow \boxed{\log Z = \sum_{n=0}^{\infty} N^{2-2h} f_h(\{\lambda_\alpha\})}$$

. Correlation Functions

Will restrict our discussion to singlet operators i.e. operators invariant under $U(N)$ symmetry

$\rightsquigarrow$ Such an operator must involve traces

$\mathrm{Tr}\,\Phi^2$, $\mathrm{Tr}\,\Phi^4$, $\mathrm{Tr}(\partial_\mu \Phi \partial^\mu \Phi)$

single-trace

$\mathrm{Tr}\,\Phi^2 \, \mathrm{Tr}\,\Phi^4 \ldots$

multi-trace

Suppose $\{O_k\}_{k=1,2,\ldots}$ denote the set of all single-trace operators, then general singlet operators can be generated from them

$\Rightarrow$ enough to restrict to correlation functions of single-trace operators.

$$\langle O_1(x_1) \cdots O_n(x_n) \rangle_c \rightsquigarrow c = \text{"connected"}$$

What is the leading order $1/N$ expansion?

There is a simple trick:

$$Z[\{J_i\}] = \int D\Phi \exp\left[iS_0 + iN\int J_i \, \mathcal{O}_i(x)\right]$$

fixed external function

$$= \int D\Phi \, e^{iS_{eff}}, \quad S_{eff} = S_0 + N\int J_i(x)\mathcal{O}_i(x)$$

$$\Downarrow$$

$$\underline{S_{eff} = N\,\mathrm{Tr}(\dots)}$$

$$G_n = \frac{1}{i^n}\frac{1}{N^n}\frac{\delta^n \log Z}{\delta J_1(x_1)\cdots\delta J_n(x_n)}\Bigg|_{J_i=0}$$

$$\Rightarrow \log Z[\{J_i\}] = \sum_{h=0}^{\infty} N^{2-2h} \cdot F_h(\{J_i, \lambda_a\})$$

$$\Rightarrow G_n = \sum_{h=0}^{\infty} N^{2-2h-n} G_n^{(h)}(\cancel{\{x_i\}})$$

without N prefactor

contribution from genus h diagrams

As $N \to \infty$, at leading order
$\langle 1 \rangle \sim O(N^2)$
$\langle \mathcal{O} \rangle \sim O(N)$
$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle \sim O(1)$
$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle_c \sim O(N^{-1})$
$\vdots$
$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_c \sim O(N^{2-n})$

Remarks:

1) $\mathcal{L} = -\frac{1}{g^2}\mathrm{Tr}\left[\frac{1}{2}(\partial\Phi)^2 + \frac{1}{2}m\Phi^2 + \frac{1}{4}\Phi^4\right]$

contains other observables which are not singlets under $U(N)$

e.g. $\Phi_b^d \Phi_d^c(x)$.

In general, such operators do not have nice scaling with N $\Rightarrow$:(

2) For YM theory, $O(N)$ symmetry is <u>local</u>. Only singlets are allowed. $\Rightarrow$ ☺

3) Almost <u>all</u> theories of interest to us are gauge theories $\Rightarrow$ ☺

4) For gauge theories, there are also nonlocal singlet operators such as Wilson loops:

$$W(C) = \mathrm{Tr}\Big(\underset{\substack{\uparrow \\ \text{path-ordering}}}{P}\exp\big(i\oint_C A\big)\Big)$$

They have the <u>same large-N scaling</u> of single-trace local operators

. The physical nature of the $N\to\infty$ limit

(a) $\langle O_i\rangle \sim O(N) \neq 0$

we include connected <u>and</u> disconnected

Variance of O: $\sigma_O^2 = \underbrace{\langle (O-\bar{O})^2\rangle_c}_{\hookrightarrow\, O(1)} = \langle O^2\rangle - \langle O\rangle^2$ ← disconnected cancels with this

$$\Rightarrow \frac{\sigma_O^2}{\langle O\rangle} \sim \frac{1}{N} \to 0 \quad \text{as } N\to\infty$$

i.e. no fluctuations.

similarly, n-point functions factorize

$$\langle O_1 \cdots O_n\rangle = \langle O_1\rangle \cdots \langle O_n\rangle + \cdots$$

is dominated by product of 1-pt functions

"classical"

(b) Redefine $\mathcal{O} \to \mathcal{O} - \bar{\mathcal{O}}$

$\Rightarrow \langle \mathcal{O} \rangle = 0$

Then to leading order in large N

$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \mathcal{O}(1)$

$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \langle \mathcal{O}_1 \mathcal{O}_2 \rangle \langle \mathcal{O}_3 \mathcal{O}_4 \rangle \cdots +$ all "contractions" $\sim \mathcal{O}(1)$

"Gaussian Theory"

is a "Generalized Free Field Theory"

$\mathcal{O}_i(t, \vec{x})$, but no com connecting $\mathcal{O}_i(t_1, \vec{x})$ with $\mathcal{O}_i(t_2, \vec{x})$

(c) Consider any connected part of the correlation functions.

$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_c \sim \mathcal{O}(N^{2-n})$

This is like a tree-level theory of interacting "particles" with coupling $1/N$

Imagine $\mathcal{O}(x)|0\rangle$ "create a single particle"

$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|0\rangle$ "two-particle"

$\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)|0\rangle$ "n-particle"

 $\sim \frac{1}{N} \sim g$,  $\sim g^2 \sim \frac{1}{N^2}$

$\langle \mathcal{O}_i \mathcal{O}_j \rangle \sim \mathcal{O}(1)$

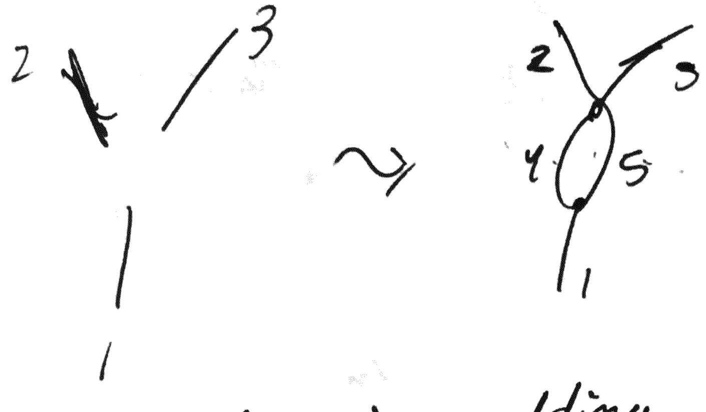
$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \sim N^{2-n} \sim g^{n-2}$ "tree-level" n-particle scattering amplitude

(d) Adding loop of $\mathcal{O}$'s:

Add intermediate states with more than one $\mathcal{O}$'s

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = \langle \mathcal{O}_1 \mathcal{O}_4 \mathcal{O}_5 \rangle \langle \mathcal{O}_4 \mathcal{O}_5 \mathcal{O}_2 \mathcal{O}_3 \rangle - N^{1} N^{-2}$$

$$\sim N^{-3}$$



Adding loop $\Rightarrow$ adding $1/N^2$

subleading terms in $1/N^2$; "loop" corrections

2.3 Strings and Matrices

$$A_n = \sum_{h=0}^{\infty} g_s^{n-2+2h} A_n^{(h)}$$

$$G_n = \sum_{h=0}^{\infty} N^{2-n-2h} G_n^{(h)}$$

scattering of strings $\longleftrightarrow$ Corr. func of single-trace operators

Topology of w.s. $\longleftrightarrow$ Topology of Feynman diagrams

$g_s \longleftrightarrow 1/N$

$A^{(h)}_{mn}$: integrate over WS of genus h $\leftrightarrow$ $G^{(h)}_n$ sum over genus h Feynman diagrams

external string $\leftrightarrow$ single-trace Φ

loops of string $\leftrightarrow$ "loops" of single-trace op

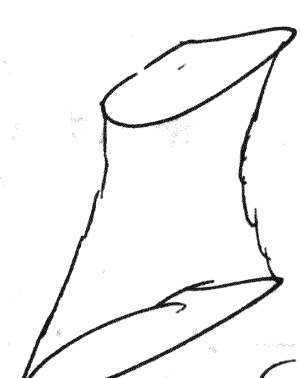
Rough argument:

$$A^{(h)}_0 = \sum_{\substack{\text{sum}\\ \text{over genus}\\ h \text{ surfaces}}} e^{iS_{string}} = \sum_{\substack{\text{triangulations}\\ \text{of genus } h\\ \text{surfaces}}} e^{iS_{string}}$$

$$\updownarrow \qquad\qquad\qquad\qquad \updownarrow$$

$$G^{(h)}_0 = \sum_{\substack{\text{genus-}h\\ \text{Feynman}\\ \text{diagram}}} \tilde{G} = \sum_{\substack{\text{triangulations}\\ \text{of genus } h\\ \text{surfaces}}} \tilde{G}$$

$W(C)$  C

$\langle W(C_1)\, W(C_2)\rangle \simeq$  C_2 C_1

~~W(C)~~

$A^a_b = {}_a\sim_b$

$W(C) = e^{i\oint_C A} \to \bigcirc$

$\Phi^a_b \Rightarrow {}_a—_b$

$\mathrm{Tr}\,\Phi^2 \Rightarrow {}_a\bigcirc_b$

$\Rightarrow$ $W(C)|0\rangle$ can be considered as <u>macroscopic</u> string states

$\leftrightarrow$ $O(x)|0\rangle$: microscopic string state

Oct 24, 2018 (Notes from Sam Leutheusser)

$A_n^{(h)}$: sum over random triangulations of a genus-h surface (and its embeddings) weighted by $e^{iS_{string}}$ | $G_n^{(h)}$: sum over triangulations of genus h surfaces weighted by E_c

Conjecture:

a duality of:

a large-N matrix-like theory $\longleftrightarrow$ a string theory

$1/N \longleftrightarrow g_s$

single trace operator $\longleftrightarrow$ single string

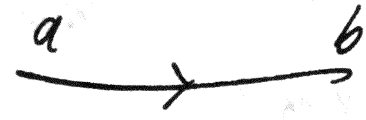
To establish this duality:

$S_{string}[\gamma_{ab}, X^\mu, \dots]$, $X^\mu(\sigma,\tau)$: Worldsheet $\to M$ "spacetime manifold"

$\rightsquigarrow$ Continuum string picture should emerge in regime where "infinitely complicated" Feynman diagrams dominate

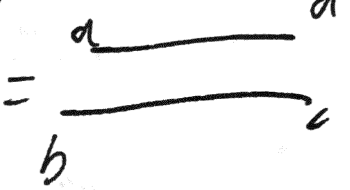
Remarks:

(1) So far, only matrix-valued fields have been considered. This can also include fields transforming in the fundamental representation of $U(N)$ $q = \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix}$ "quarks"

$\langle q^a q_b \rangle =$ 

Precise mapping to a string theory with both closed and open strings (i.e. quarks add open strings)

(2) In addition to $U(N)$, can also consider $SO(N)$, $Sp(N)$

$\langle \Phi_{ab} \Phi_{cd} \rangle =$ 

(no arrows) $\Rightarrow$ include non-orientable surfaces

$\rightsquigarrow$ maps to non-orientable string theory

<u>Explicit example:</u> (0-dimensional)

$$e^{-Z} = \int dM \exp\left[-\frac{N}{g} \mathrm{Tr}[V(M)]\right]$$

M = hermitian matrix

$$V(M) = \frac{1}{2}M^2 + \sum_{k=3} a_k M^k$$

$\sim Z = Z_0 + Z_1 + \cdots$

$Z_0 \sim O(N^2)$

$$dM = \prod_{a,b} dM^a{}_b$$

(for $a=b$ this is the usual $dM^a{}_b \in \mathbb{R}$

for off-diagonal $dM^a{}_b = dM^a{}_b\, dM^a{}_b$)

Since $Tr(V(M))$ depends only on eigenvalues,
write $M = U^+ \Lambda U$, $\Lambda = diag(\lambda_1 \dots \lambda_N)$

$\rightsquigarrow$ Measure might depend on λ_i

$\Rightarrow Tr(V(M)) = \sum_{i=1}^{N} V(\lambda_i)$, $dM = \prod_{i=1}^{N} d\lambda_i \, \Delta^2(\Lambda)$ "DU"

It turns out $\Delta(\Lambda) = \prod_{i<j} (\lambda_i - \lambda_j)$ (Vandermonde determinant)

shown in problem set

$$\Rightarrow e^{-Z} = \int \prod_i d\lambda_i \, \Delta^2(\Lambda) \, e^{-\frac{N}{g} \sum_i V(\lambda_i)} \leftarrow \underline{N\text{-sum} \Rightarrow \underline{O(N^2)}}$$

$\rightsquigarrow$ Naive saddle point: $V'(\lambda) = 0$ (incorrect)

$$\Delta^2(\Lambda) = \exp\Big[\underbrace{\sum_{i<j}}_{\text{double sum} \Rightarrow O(N^2) \text{ too}} \log(\lambda_i - \lambda_j)^2\Big] \leftarrow \text{cant have } \lambda_i = \lambda_j$$

as $\log \to -\infty$
"level repulsion"

$\Delta^2(\Lambda)$: 1) $O(N^2)$

2) Repulsion between λ_i s

$\Rightarrow \lambda_i$ cannot all sit at minimum

EOM for λ_i:

$$2\sum_{j\neq i} \frac{1}{\lambda_i-\lambda_j} = \frac{N}{g} V'(\lambda_i) \quad (*)$$

(get this by directly differentiating)

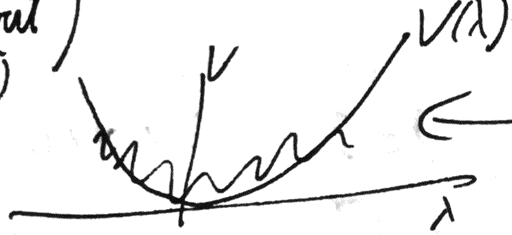
$N\to\infty$ limit (expect $\rho(\lambda)$ to form a continuous function)

$$\int_{-\infty}^{\infty} \rho(\lambda)\, d\lambda = 1, \quad \rho(\lambda) = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \delta(\lambda-\lambda_i)$$

$$\Rightarrow \boxed{2P\int d\lambda' \frac{\rho(\lambda')}{\lambda-\lambda'} = \frac{1}{g}V'(\lambda)} \quad (**)$$

principal value (i.e. throw out $i=j$)

Note:



λ's pushed out, but not to ∞ as this would cause infinite energy so as $N\to\infty$, we get continuous distribution of λ_i (~~not~~ infinite range)

Now assume $\rho(\lambda)$ only supported on finite interval in $\mathbb{R}$, denoted by I

Introduce:

$$F(\xi) = \int_I d\lambda' \frac{\rho(\lambda')}{\xi-\lambda'} \quad \text{for general complex } \xi$$

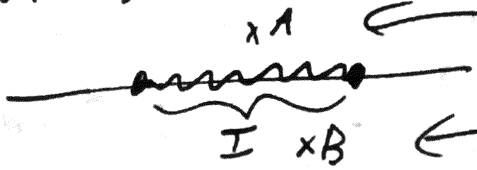
Then $(**)$ is equivalent to: $P\, F(\xi) = \frac{1}{2g}V'(\lambda)$ at $\xi = \lambda - i\varepsilon$ for $\lambda\in I$

The preceding equation comes from the identity:

$$\frac{1}{x \pm i\epsilon - \lambda'} = P\frac{1}{x-\lambda'} \mp i\pi\,\delta(x-\lambda') \qquad (***)$$

$F(\xi)$ satisfies the following analyticity properties:

(1) Analytic ~~properties~~ function on complex plane except for branch cut at I



compare A and B values using $(***)$ to see branch cut

(2) On real axis, $F(\xi)$ is real for $x \notin I$

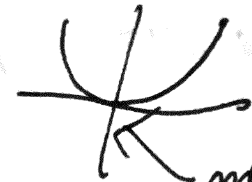
(3) $F(\xi) = \frac{1}{\xi} + \dots$ as $\xi \to \infty$

since for $\xi \notin I$, $\frac{1}{\xi - \lambda'} \to \frac{1}{\xi}\frac{1}{1-\lambda'/\xi}$, $\int \rho = 1$

(4) $\mathrm{Re}(F(x-i\epsilon)) = \frac{1}{2g}V'(x)$

$\mathrm{Im}(F(x-i\epsilon)) = \pi\rho(x)$

These conditions determine $F(\xi)$ (and therefore $\rho(\lambda)$) completely

<u>Example</u>: $V(\lambda) = \frac{\lambda^2}{2} + \lambda^4$,  min at 0

take $I = [-a, a]$, a to be determined

write ansatz for $F(\xi)$:

$$F(\xi) = \frac{1}{2g} V'(\xi) + f(\xi)\sqrt{\xi^2 - a^2}$$

, f is TBD

$\leftarrow$ pure imaginary for $\xi \in I$

(3) + analyticity gives $f(\xi) = \frac{-1}{2g}(1 + 4\lambda^2 + 2a^2)$

$$a^2 = \frac{1}{6}\left(\sqrt{1+48g} - 1\right)$$

$$\Rightarrow \rho(\lambda) = \frac{1}{2\pi g}\left(4\lambda^2 + 1 + 2a^2\right)\sqrt{a^2 - \lambda^2} \quad \leftarrow \lambda \in [-a, a]$$

$$\Rightarrow Z_0 = N^2\left[-\frac{1}{g}\int_{-a}^{a} d\lambda\, \rho(\lambda) V(\lambda) - P\int_{-a}^{a} d\lambda\, d\mu\, \rho(\lambda)\rho(\mu)\log|\lambda - \mu|\right]$$

from potential

from Vandermonde

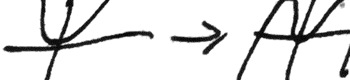
Maps to string theory in 0 dimensions!

Oct 29 (Notes from Sam Leutheusser)

Remarks

(0) $g \to 0$, find $\rho(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}$ ← Wigner distribution for M gaussian

(1) $Z_0[g]$ is analytic in g for small g. When expanded as a power series, there is a finite radius of convergence

⇝ If g flips sign, one would expect 

⇝ essential sing. at $g=0$ (theory is unstable)

(comes from total # of feynman diagrams $\sim n!$)

But, planar diagrams are polynomial and contribute most at large N

(2) a^2 has branch point in g at $g = g_c = -1/48$ ⇒ radius of conv. $= 1/48$

Perturbation theory breaks down there

near g_c, $Z_0[g] \sim$ analytic in $g + \alpha\,(g_c - g)^{5/2} \ldots$

planar diagram singularity

(3) From perspective of summing planar diagrams, to see this non analytic behavior one must sum full series (all powers of g)

$$(g_c - g)^{5/2} \sim \sum_n \underbrace{n^{-7/2}}_{\text{non analytic behavior}} \left(\frac{g}{g_c}\right)^n \leftarrow \text{at large } n$$

(4) Only in $g \to g_c$ limit can we expect a continuum description of string theory to emerge (need arbitrarily complicated triangulations for continuum)

(5) Consistency check: all Z_h $\forall h$ need non-analytic behavior at $g = g_c$ for continuum limit to be string theory

$Z_h[g]$ non-analytic at $g = g_c = -1/48$ $\forall h$ ✓

near g_c, $Z_n[g] \sim |g-g_c|^{\frac{\chi}{2}(2-\Gamma)}$ $\quad$ $\chi = 2-2g$ (Euler char.) $\quad$ $\Gamma = -1/2$ in this case

(6) String theory dual

$X: (\sigma,\tau) \to M$, γ_{ab}, φ: internal worldsheet d.o.f.

0-d string theory: M is a point, X is trivial

$$Z_{string} = \int D\gamma_{ab}\, D\Phi\, e^{-S_{string}[\gamma,\Phi]}$$ spacetime is a point, so work in Euclidean picture

$$S_{string}[\gamma,\varphi] = \mu \underbrace{\int d^2\sigma \sqrt{\gamma}}_{\text{Area, } A} + \lambda \underbrace{\frac{1}{4\pi}\int d^2\sigma \sqrt{\gamma} R}_{\text{Euler number, } \chi = 2-2h} + S_{matter}[\Phi]$$

$$\Rightarrow S_{string} = \mu A + \lambda \chi + S_m[\Phi]$$

Identify $\frac{1}{N} \sim e^{\lambda}$ (controls expansion in genus)

$\mu \propto \underbrace{(g-g_c)}_{\text{only param}}^{\#}$ $\leftarrow$ power ends up being 1 since μ is chemical potential for area: $\frac{\partial}{\partial\mu} Z = \langle A \rangle$

as $g \to g_c \Rightarrow n$ in expansion of $(g-g_c)^{n}$ is $\simeq$ area

$$\rightsquigarrow Z_{string} = \sum_h \int D\gamma\, D\Phi\, e^{-\mu A - \lambda\chi - S[\Phi]}$$

After gauge fixing, reparam. invariance of γ allows us to take φ to be the remaining d.o.f. of γ

$$\Rightarrow Z_{string} = \sum_h \int D\varphi\, D\Phi\, \exp[-S_L(\varphi) - S_{matter}(\Phi)]$$

Here $S_L = \int d^2\sigma \underbrace{\sqrt{\hat{g}_0}}_{\text{gauge fixed}} [(\partial\varphi)^2 + \varphi \cdot \underset{\uparrow \text{ parameter}}{Q} R + \mu e^{2\gamma\varphi}]$

→ φ behaves like an inhomogeneous "emergent" spatial dimension. Even though X^μ is trivial, we grow a dimension.

$$Z_h^{(\text{string})} \propto \mu^{\frac{\chi}{2}(2-\Gamma_{\text{string}})}$$

For S_{matter} trivial, we find $\Gamma_{\text{string}} = -1/2$.
(matches matrix theory)

2.3 String Description of a ~~ga~~ Gauge theory

Non-Abelian $SU(N)$ gauge theory $\Leftrightarrow$ String theory?
$N \to \infty$ in d-dim

- A simplest guess: Maybe a string theory in $\underbrace{\text{Mink}_d}_{ds^2 = -dt^2 + d\vec{x}^2}$

→ This does not work!

(1) Such a string theory appears inconsistent
A string theory in Mink_d is consistent only for $d=26$ (bosonic), $d=10$ (superstring)

(2) ~~A~~ How about $\text{Mink}_4 \times N$ ← compact
so string theory has $SO(3,1)$ symmetry

But all string theories on $\text{Mink}_4 \times N$ have 4d graviton → violates Weinberg-Witten
W-W applies because $SU(N)$ theory has gauge-inv conserved stress tensor.

→ We'd want a string theory "without gravity"?

→ impossible!

(Hints:

(1) φ with Mink_4 combines to form 5d curved spacetime

→ 5d non-compact spacetime

→ 5d gravity

→ Does not contradict Weinberg-Witten

(2) Holographic Principle

4d gauge theory could in principle be related to 5-d gravity

→ One could try a string theory in $Y_5 \times M$

non-compact spacetime (Y_5)

compact space (M)

$\Rightarrow Y_5$ should have all the Mink_4 symmetries

$$\rightsquigarrow ds^2 = g(z)\left(-dt^2 + d\vec{x}^2\right) \underset{f(z)}{+} dz^2$$

$$= \Omega^2(z)\left(-dt^2 + d\vec{x}^2 + dz^2\right) \quad (*)$$ after redefinition

Suppose the 4-d theory is scale-invariant

$$(**) \quad (t, \vec{x}) \to \lambda(t, \vec{x})$$

(*) must be invariant under (**)

$\Rightarrow$ must have $z \to \lambda z$, ~~$\Omega(z) \to \Omega(\lambda z)$~~

$$\Omega(\lambda z) = \frac{1}{\lambda}\Omega(z)$$

This fixes $\Omega(z) = c/z$

So, for scale-invariant lower dimensional theory, we would have

$$ds^2 = \frac{R^2}{z^2}\left[-dt^2 + d\vec{x}^2 + dz^2\right]$$

R is some constant

This is AdS.

AdS_5 has isometry group $SO(2,4)$

88

Last time:

$$\underbrace{Y_5}_{\text{non-compact}} \times \underset{\sim}{N}$$

does not contradict Weinberg-Witten

$$ds^2 = \Omega^2(z)(-dt^2+dx^2+dz^2)$$

1997 Polyakov wrote this down to study QCD.

Nov. 1997 Maldacena wrote this, realizing AdS

scale-invariant: $(t,\vec{x}) \to \lambda(t,\vec{x})$, $z \to \lambda z$ $\Rightarrow \Omega(\lambda z) = \frac{1}{\lambda}\Omega(z)$

$\Rightarrow \Omega(z) = \frac{R}{z}$ (R: const)

$$\Rightarrow \underbrace{ds^2 = \frac{R^2}{z^2}(-dt^2+d\vec{x}^2+dz^2)}_{AdS_5 \text{ space}}$$

determined uniquely (up to R)

scale invariance $\rightsquigarrow$ conformal invariance $\underbrace{SO(2,4)}_{\text{isometry}}$

· Outline of string theory & derivation of AdS/CFT

(a) closed strings

quantization of a closed string in a fixed spacetime:

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{-\gamma}\,\gamma^{ab}\partial_a X^\mu \partial_b X_\mu \quad (*)$$

$$X^\mu(\sigma,\tau) = X^\mu(\sigma+2\pi,\tau)$$

Tricky to quantize, but the procedure is well-known

⇒ String ~~theory~~ excitation spectrum

(1) Not all spacetime allow a consistent string propagation quantum mechanically

~~(1)~~ For (*), we have bosonic string theory:

<u>Mink</u>: $d=26$

Taking $(\underset{\text{super coordinates}}{X^\mu, \psi^\mu})$: superstring

Mink: $d=10$

~~(3)~~ (2) Spectrum:

Oscillation excitation of a string ↔ spacetime particle

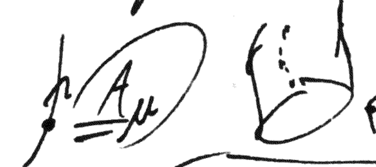
<u>Massless</u>: $\underbrace{h_{\mu\nu}, B_{\mu\nu}, \Phi}_{\text{universal to all string theories}}, \ldots$

<u>Massive</u>: $m^2 = N/\alpha'$

(infinite towers of massive modes)

$\alpha' = l_s^2$ $\quad l_s$: string length

$h_{\mu\nu}$: massless spin-2 (graviton)

$\underbrace{B_{\mu\nu} = -B_{\nu\mu}}_{\text{antisymmetric tensor}}$:  ↔ $B_{\mu\nu}$ ~~α~~ "gauge field" for a string ($U(1)$)

Φ: $g_s = \langle e^{\Phi} \rangle$ $\leftarrow$ $\boxed{g_s \text{ can vary spatially if } \Phi \text{ does}}$ coupling of the string

g_s ← "free parameter"

$G_N \propto g_s^2 \Rightarrow G_N = g_s^2 \alpha'^{\frac{d-2}{2}}$

At low energies: $E^2 \ll 1/\alpha'$

effective theory: Einstein gravity + matter (massless)

+ higher derivative corrections

Due to presence of graviton:

spacetime metric becomes dynamical

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

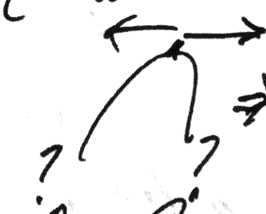
$\Rightarrow$ closed strings must themselves be excitations of spacetime

(* this logic is still not clear to me)

(3) Bosonic string:

there exists excitation with negative N

i.e. $m^2 < 0$ $\Rightarrow$ unstable



Superstrings can be stable

open question

5 different perturbative superstrings

We now know they are related nonperturbatively

Their spectra contain spacetime fermions

At low energies: Supergravity

(interesting note: there are "other" superstring theories with only bosons but emergent fermions.)

Massless: $(d=10)$

IIA: $h, B^{(2)}, \Phi, C_\mu, C^{(3)}_{\mu\nu\lambda}$, + fermions

IIB: $h, B^{(2)}, \Phi, \chi, C^{(2)}_{\mu\nu}, \underset{\sim}{C^{(4)}_{\mu\nu\lambda\rho}}$ + fermions

$C^{(2)}, C^{(3)}, C^{(4)}$ are fully anti-symmetric

$C^{(4)}$: self-dual

$F^{(5)} := dC^{(4)}, \quad F^{(5)} = * F^{(5)}$

(b) Open string

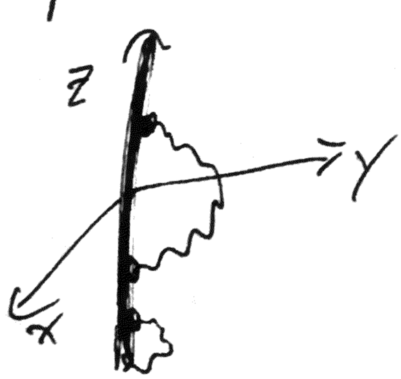
need to impose boundary conditions at ~~the~~ end points

They can only end on some "special places" which can be considered as some kind of "defects" in spacetime

"D-branes"

a D_p-brane has p spatial dimensions

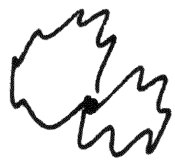
D1-brane



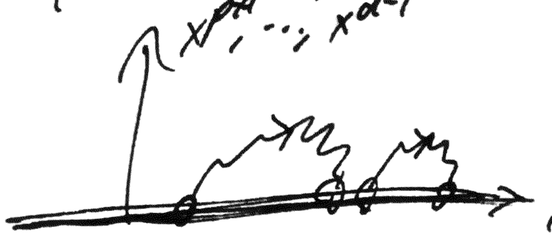
D3-brane "spacetime-filling"
$\rightsquigarrow$ can end anywhere

Classically, D-brane can be considered as a proxy of specifying boundary conditions on an open string

<u>D0-brane</u>



Given a D-brane configuration, you can quantize open string ending on it.



quantize open strings:

<u>massless</u>: $A_\mu(x^\nu)$

$\mu, \nu = 0, \dots, p$

$\Phi^i(x^\mu) \quad i = p+1, \dots d-1$

<u>massive</u>: $m^2 = N/\alpha'$

endpoint of a string: charged particle

A_μ: gauge field for the charge

Φ^i transverse motion of the brane <u>itself</u>

$\Rightarrow$ D-branes are dynamical objects

$\Rightarrow$ open strings are excitations of D-branes

Some D-branes have $N<0$ $\Rightarrow$ unstable excitations

$\rightsquigarrow$ decay into closed string modes

There are stable-situations

IIA $\quad 0, 2, 4, 6, (8) \quad$ (even-dimensional)

IIB $\quad (-1), 1, 3, 5, 7, (9) \quad$ (odd-dimensional)

An object is stable if it carries some sort of "gauge" charge.

A p-dimensional object can carry charge of $(p+1)$-form

Nov 5, 2018

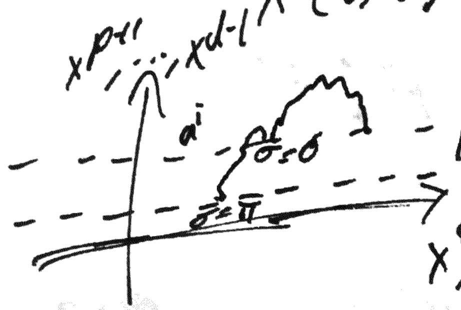
$\sigma \in [0, \pi]$

$X^\mu(\sigma, \tau) \rightsquigarrow X^\mu(0,\tau) \Rightarrow \begin{cases} a^\mu & \text{Dirichlet} \\ \partial_\sigma X^\mu \big|_{\sigma=0} = 0 & \text{Newman} \end{cases}$

"no momentum exits from the string"

$\underline{\sigma = 0}$ $\quad X^\mu(0,\tau) \quad \mu = 0, 1, \cdots p \quad$ Newman

$X^i(0,\tau) = a^i \quad i = p+1, \cdots, d-1 \quad$ Dirichlet



Massless excitations: $\quad A_\mu(x^\nu) \quad \mu, \nu = 0, 1, \cdots p$

$\Phi^i(x^\mu) \quad i = p+1, \cdots, d-1$

Massive: $\quad m^2 = N/\alpha' \quad \leftarrow$ again can be negative $\Rightarrow$ unstable D-branes

$\Rightarrow$ D-branes are dynamical objects

$\Rightarrow$ Open strings are excitations of D-branes

Stable D-branes:

IIA: even dimensions

0, 2, 4, 6, (8)

IIB: odd-dimensional ones

(-1), 1, 3, 5, 7, (9)

$$h_{\mu\nu},\ B_{\mu\nu},\ \Phi + \begin{cases} C^{(1)}_{\mu},\ C^{(3)}_{\mu\nu\lambda} & \text{IIA} \\ \chi,\ C^{(2)}_{\mu\nu},\ C^{(4)}_{\mu\nu\lambda\rho} & \text{IIB} \end{cases}$$

Massless closed string spectrum

Anti-symmetric potentials are generalization of $U(1)$ Maxwell field to higher forms

$$A_\mu \qquad A = A_\mu dx^\mu$$

$$F = dA \qquad \mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

$$C^{(n)} = C_{\mu_1 \cdots \mu_n}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}$$

$$F^{(n+1)} = dC^{(n)} \qquad \mathcal{L} = -\frac{1}{2}\frac{1}{n!}(F \wedge \star F)$$

$$C^{(n)} \to C^{(n)} + d\Lambda^{n-1}$$

$\Rightarrow$ point charged particle with path x^μ couples to A as $\int_C A_\mu \frac{dx^\mu}{d\tau} d\tau = \int_C A$ ← pullback to particle worldline

p-dimensional object has $(p+1)$-world volume Σ,

$$\int_\Sigma C^{(p+1)} = \int_\Sigma d^{p+1}\xi\ C^{(p+1)}_{\mu_1 \cdots \mu_{p+1}} \frac{\partial X^{\mu_1}}{\partial \xi^1} \cdots \frac{\partial X^{\mu_{p+1}}}{\partial \xi^{p+1}}$$

(← pullback of $C^{(p+1)}$ to Σ)

with $X^\mu(\xi)$ denoting embedding of Σ in spacetime

For any $C^{(n)} \to F^{(n+1)} = dC^{(n)} \to \tilde{F}^{(d-n-1)} = \star F^{n+1}$

$$\tilde{F}^{d-n-1} = d\tilde{C}^{(d-n-2)}$$

Given $C^{(n)}$ there is a $p = n-1$ dimensional object charged under "electric"

and a $d-n-3$ dimensional object charged under "magnetic"

$B_{\mu\nu}$: String — electric

NS5-brane — magnetic

IIA: $C^{(1)}$ — D0-brane (E)

D6-brane (M)

$C^{(3)}$ — D2-brane (E)

D4-brane (M)

IIB: χ — D(-1)-brane (E)

D7-brane (M)

$C^{(2)}$ — D-string (E)

D5-brane (M)

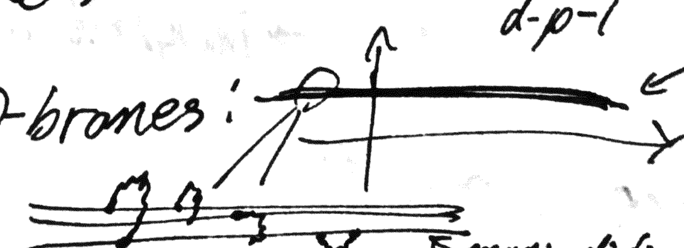
$C_+^{(4)}$ — D4 brane (EM)

Properties of D-branes:

- At low energies

A single D-brane ⇒ U(1) Maxwell + Φ^i (Free scalar) in $d-p-1$

N coincidental D-branes:



← N D-branes right on top of one another

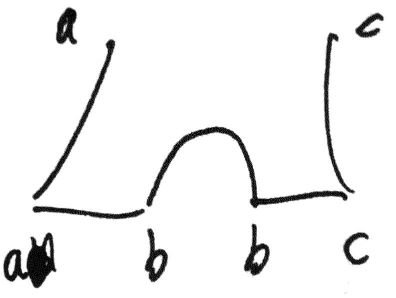
← many distinct strings

⇝ Each string has two endpoints ⇒ two labels $1, \dots, N$

⇒ massless Fields $(A_\mu)^a{}_b$ $\quad a,b = 1, \dots, N$

$(\Phi^i)^a{}_b$ ← you can show these are hermitian

⇒ $U(N)$



$\Rightarrow$ matrix product

low-energy limit:

$$S = -\frac{1}{g_{YM}^2}\int d^{p+1}x \; Tr\left(\frac{1}{4}F^2 + D_\mu \Phi^i D^\mu \Phi_i + [\Phi^i, \Phi^j]^2\right)$$

$+ \cdots$ (fermionic terms)

$F_{\mu\nu} = \partial_{[\mu} A_{\nu]} - i[A_\mu, A_\nu]$

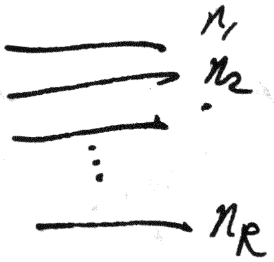
$D_\mu \Phi^i = \partial_\mu \Phi^i - i[A_\mu, \Phi^i]$ $\qquad g_{YM}^2 \sim g_s$

$\boxed{\text{Maximally supersymmetric YM}}$

D3-brane: $\Rightarrow p+1 = 4$ $\qquad \mathcal{N}=4$ SYM

When we separate the branes:

$\langle \Phi^i \rangle^a{}_a \neq 0$



$U(N) \rightarrow U(n_1) \times U(n_2) \cdots \times U(n_R)$

• D-branes gravitate

a charged point particle in $D=4$

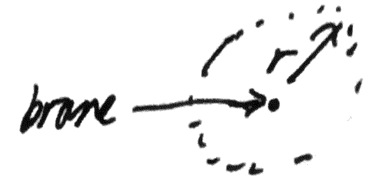
$$S = \int d^4x \sqrt{-g}\left[\frac{1}{16\pi G_N} R - \frac{1}{4}F^2\right] - m\int_C ds + q\int A$$

$\partial_\mu F^{\mu\nu} = j^\nu$ (particle)

$G_{\mu\nu} = 8\pi G_N \left(T_{\mu\nu}(\text{particle}) + T_{\mu\nu}(EM)\right)$

E.g. Consider a D3-brane

$\underline{1,2,3}$ $\quad$ $\underline{4,5,6,7,8,9}$



N ~~D~~D3-branes

$$ds^2 = f(r)\left(-dt^2 + d\vec{x}^2\right) + g(r)\left(dr^2 + r^2 d\Omega^2\right)$$

$$f^{-1} = g = \left(1 + \frac{R^4}{r^4}\right)^{1/2}, \quad R^4 = 4\pi g_s (\alpha')^2 N \propto N T_3$$

tension of D3-brane

$r=0 \to$ location of D3-brane

$r \to \infty \to$ "Minkowski space"

R characterizes when gravitational effect becomes strong.

note that Schwarzschild metric has a $-$ sign

At $r \to 0$, $f(r) = \dfrac{r^2}{R^2}$, $g(r) = \dfrac{R^2}{r^2}$

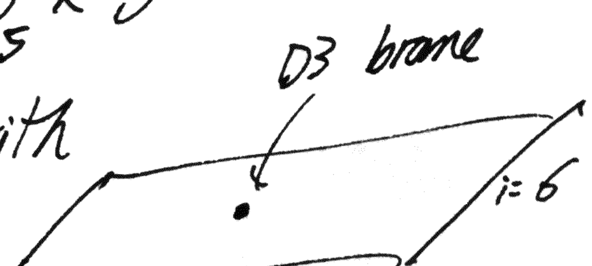
$$\Rightarrow ds^2 = \frac{r^2}{R^2}\left(-dt^2 + d\vec{x}^2\right) + \frac{R^2}{r^2}\left(dr^2 + r^2 d\Omega_5^2\right)$$

$$= \underbrace{\frac{r^2}{R^2}\left(-dt^2 + d\vec{x}^2\right) + \frac{R^2}{r^2} dr^2}_{AdS_5} + R^2 d\Omega_5^2$$

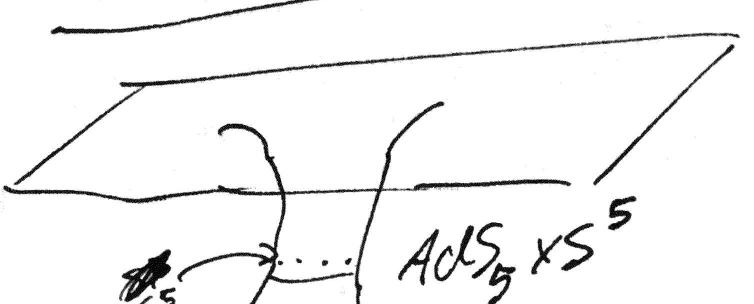
S^5 never shrinks to 0 size (const radius $= R$)

$$AdS_5 \times S^5$$

Started with



Became:



$$\frac{dr^2}{r^2} = dl^2$$

$$l = \log r$$

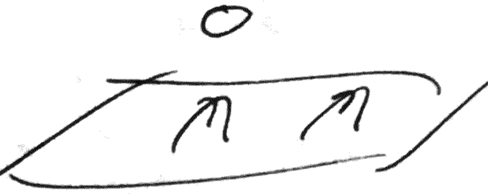
($r=0$ sits an infinite proper distance away)

100

Nov 7, 2018 (Notes from MIT Open Courseware Fall 2014 8.821 Lecture 16)

We thus have two descriptions of D3-branes

(A) D-branes in flat Mink_{10} where open strings can live



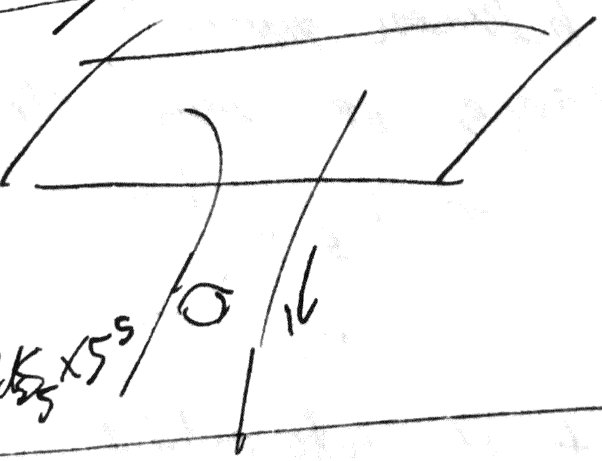
$\mathbb{R}^{3,1}$

(B) Using spacetime metric:

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + d\vec{x}^2) + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2$$

($+ F_5$ flux on S^5)

$\rightsquigarrow$ only closed strings that see a curved geometry



Mink_{10}

$AdS_5 \times S^5$

A = B

Both descriptions can in principle be valid for all α' and g_s

Maldacena (1997)

low energy limit ⇒ AdS/CFT

What is the low energy limit?

Fix E, take $\alpha' \to 0$
(fix α', $E \to 0$) $\rightsquigarrow \alpha' E^2 \to 0$

(A): Open string sector ⇒ $\mathcal{N}=4$ Super Yang-Mills with gauge group $U(N)$

$$g_{YM}^2 = 4\pi g_s$$

Closed string sector ⇒ graviton, dilaton

couplings between massless closed and open strings, or closed strings themselves

$$G_N \propto g_s^2 \alpha'^4$$

$$E \to 0 \Rightarrow G_N E^8 \to 0$$

So we get the interacting $\mathcal{N}=4$ theory + free massless modes as $E \to 0$

(B): As before:

$$ds^2 = f(r)(-dt^2 + d\vec{x}^2) + g(r)(dr^2 + r^2 d\Omega_5^2)$$

$$f^{-1} = g = \left(1 + \frac{R^4}{r^4}\right)^{1/2}$$

Curved spacetime: must be careful to specify what "energy" to use

E in (A): defined w.r.t. t (i.e. time at $r = \infty$)

At r: local proper time

$$d\tau = g^{-1/2} dt$$

$$\Rightarrow E_\tau = g^{1/2} E$$

For $r \gg R$: $g \sim 1$

$E^2\alpha' \to 0 \Rightarrow$ all massive closed strings decouple

For $r \ll R$ $\quad g \sim \frac{R^2}{r^2}$

$$E^2\alpha' \to 0 \Rightarrow E_\tau^2 \frac{r^2}{R^2}\alpha' \to 0$$

$$\Rightarrow E_\tau^2 \frac{r^2}{\sqrt{4\pi g_s N}} \to 0$$

$\Rightarrow$ For any E_τ, low energy limit has $r \to 0$

So Low energy limit gives free gravitons at $r = \infty$ (Mink_{10})
+ full string theory in $AdS_5 \times S^5$
(with flux)

$$A \quad = \quad B$$

$\downarrow$ $\qquad\qquad$ $\downarrow$ (low energy limit)

$\mathcal{N}=4$ SYM theory + free graviton $\qquad$ IIB String in $AdS_5 \times S^5$ + free graviton

$$\Rightarrow \boxed{\begin{array}{ccc} \mathcal{N}=4 \text{ SYM theory with gauge group } U(N) & = & \text{IIB String in } AdS_5 \times S^5 \end{array}}$$

Wednesday Nov. 14, 2018

$\mathcal{N}=4$ SYM theory with gauge group $U(N)$ in $M_4 = (1,3)$

is

$\mathbb{II}$B String theory in AdS

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + dx^2) + \frac{R^2 dr^2}{r^2} + R^2 d\Omega_5^2$$

plus F_5^+

$$g_{YM}^2 = 4\pi g_s \rightsquigarrow \lambda = g_{YM}^2 N = \frac{R^4}{(\alpha')^2}$$

$$\underbrace{16\pi G_N}_{\text{fundamental}} = (2\pi)^7 g_s^2 (\alpha')^4 \Rightarrow \frac{G_N}{R^8} = \frac{\pi^4}{2N^2}$$

2.5 Anti-de Sitter spacetime AdS_{d+1}

Homogenous spacetime of constant negative curvature. Consider hyperboloid in $(2,d)$

$$X_{-1}^2 + X_0^2 - \sum_{i=1}^{d} X_i^2 = R^2$$

with metric $ds^2 = -dX_{-1}^2 - dX_0^2 + \sum_{i=1}^{d} dX_i^2$

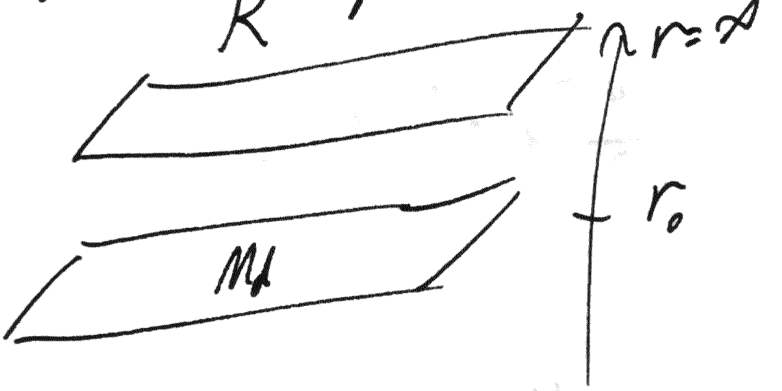
$\rightsquigarrow SO(2,d)$ isometry

(i) Poincare coordinates

$r = X_{-1} + X_d, \quad x^\mu = R\frac{X^\mu}{r} \quad \mu = 0, \dots, d$

$r > 0$

$$\Rightarrow ds^2 = \frac{r^2}{R^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{R^2}{r^2}dr^2$$



(ii) Global coordinates take $X_{-1} = R\sqrt{1+\rho^2}\cos t \quad X_0 = R\sqrt{1+\rho^2}\sin t$

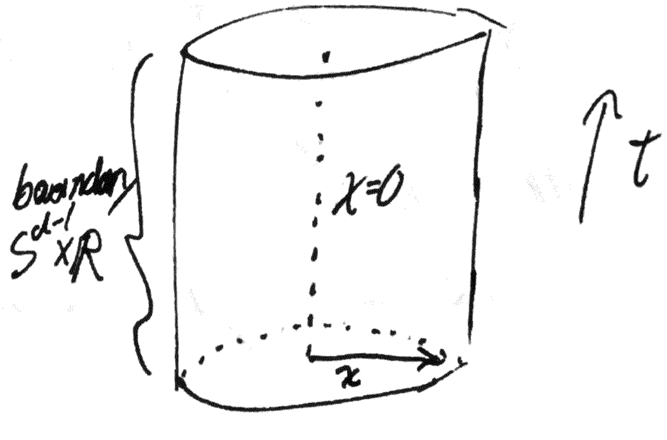
$\sum_{i=1}^{d} X_i^2 = R^2\rho^2 \quad \rightsquigarrow \quad X_{-1}^2 + X_0^2 = R^2(1+\rho^2)$

$$\Rightarrow ds^2 = R^2\left[-(1+\rho^2)dt^2 + \frac{d\rho^2}{1+\rho^2} + \rho^2 d\Omega_{d-1}^2\right]$$

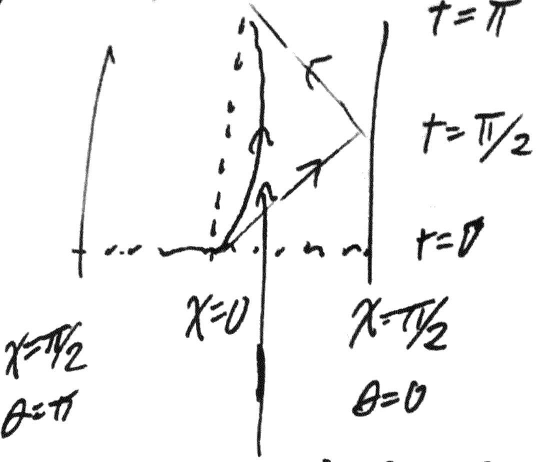
$\rho = \tan\chi \quad (\chi \in [0, \pi/2))$

$$ds^2 = \frac{R^2}{\cos^2\chi}\left[-dt^2 + d\chi^2 + \sin^2\chi \, d\Omega_{d-1}^2\right]$$

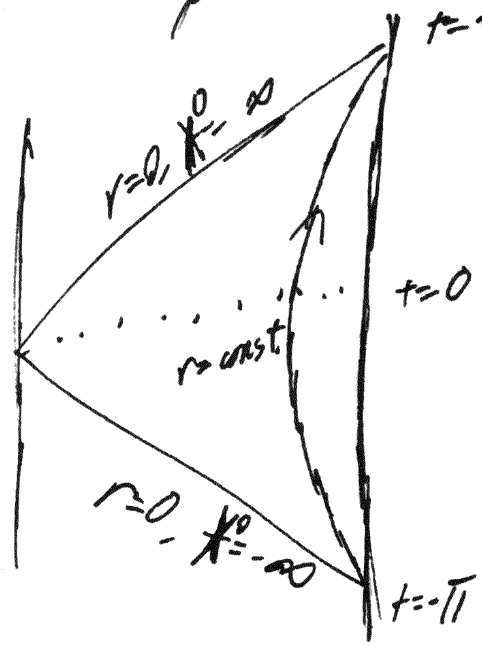
The causal structure of AdS_{d+1} is that of a solid cylinder: $B^d \times R$

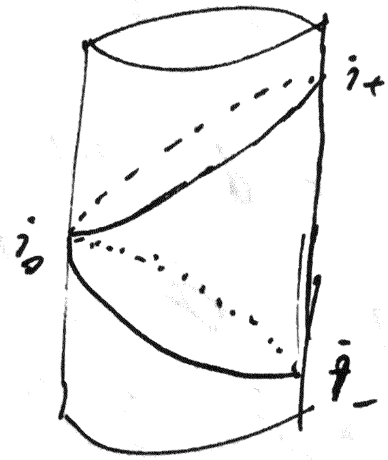


Light ray can reach the boundary in finite time



a massive particle cannot. It will be pulled back by gravitational pull.





global AdS contains infinite # of copies of Poincare patch.

· Symmetries of AdS_{d+1}

isometry: $SO(2,d)$, $\frac{1}{2}(d+2)(d+1)$ generators

$$ds^2 = \frac{r^2}{R^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{R^2}{r^2}dr^2 = \frac{R^2}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right) \quad (*)$$

P^μ: Translation along x^μ $\quad d$

$M^{\mu\nu}$: Lorentz trans. for x^μ $\quad \frac{1}{2}d(d-1)$

scaling: $z \to \lambda z$, $x^\mu \to \lambda x^\mu$

(*) is also invariant under

$$I = \begin{cases} z \to \dfrac{z}{z^2+x^2} \\ x^\mu \to \dfrac{x^\mu}{z^2+x^2} \end{cases}$$ not connected to identity

$\rightsquigarrow$ special conformal: $I \circ p^\mu(b_\mu) \circ I$ d

$$z' = \frac{z}{1+2b\cdot x+b^2(z^2+x^2)}$$

$$x^{\mu\prime} = \frac{x^\mu + b^\mu(x^2+z^2)}{1+2b\cdot x+b^2(z^2+x^2)}$$

$\frac{1}{2}(d+2)(d+1) = d + \frac{1}{2}d(d-1) + 1 + d$ ✓

. Symmetries of S^n: $SO(n+1)$

$\rightsquigarrow$ $AdS_5 \times S^5$: $SO(2,4) \times SO(6)$

String Theory in $AdS_5 \times S^5$

g_s, α', R

$\rightsquigarrow$ Two dimensionless parameters

g_s, $\frac{\alpha'}{R^2}$ i.e. $\left(\frac{G_N}{R^8}, \frac{\alpha'}{R^2}\right)$

. classical gravity as $\underset{\substack{\uparrow \\ \text{string weakly interacting}}}{g_s \to 0}$, $\underset{\substack{\uparrow \\ \text{massive modes decouple}}}{\frac{\alpha'}{R^2} \text{ " } l_s^2/R^2 \text{ " } \to 0}$

$\Rightarrow$ IIB supergravity

= Einstein gravity + finite # of matter fields

. "Classical" string limit

$$\frac{\alpha'}{R^2} = \text{finite}, \quad g_s \to 0$$

$$\varphi(x^\mu, z, \Omega_5) = \sum_\ell \underbrace{\varphi_\ell(x^\mu, z)}_{\substack{\uparrow \\ \text{fields in} \\ AdS_5}} \underbrace{Y_\ell(\Omega_5)}_{\substack{\uparrow \\ \text{harmonics} \\ \text{on } S^5}}$$

$\Rightarrow$ 5-dimensional gravity

$$S_{gravity} = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g}\, R_5 + \text{matter}$$

$$\Rightarrow G_5 = \frac{G_N^{(10)}}{\underset{\substack{\uparrow \\ \text{Volume of } S^5}}{V_5}} = \frac{G_N}{\pi^3 R^5}$$

• $\underline{\mathcal{N}=4 \text{ SYM } (3+1)}$

$\underline{\text{Field content}}$: A_μ, φ^i $(i=1,\dots,6)$, χ^A_α $(A=1,\dots,4)$

all in adjoint rep'n of $U(N)$

$\sim$ all $N\times N$ hermitian matrices

altogether: $(8b + 8f) \times N^2$

on-shell d.o.f.

Interacting part: $SU(N)$

$U(1)$ part: Free

$$\mathcal{L} = -\frac{1}{g_{YM}^2} \mathrm{Tr}\left(\frac{1}{4}F^2 + \frac{1}{2}(D_\mu \varphi^i)(D^\mu \varphi^i) + [\varphi^i, \varphi^j]^2 + \text{fermionic part}\right)$$

Properties:

(1) Has $N=4$ supersymmetries

Susy: boson $\leftrightarrow$ fermion

$\Rightarrow$ conserved fermionic charges

trans parameter: spinor (Weyl)

4 such indep. spinor parameters

The "simplest strongly-interacting 4-D theory"

(2) g_{YM} is a dimensionless quantity

<u>and</u>

β-function is 0

(3) Conformally-invariant.

continuing: (Nov 19, 2018)

$$\text{IIB String theory in } AdS_5 \times S^5 = \mathcal{N}=4 \text{ SYM with } U(N) \text{ in Mink}_4$$

$\qquad\qquad\|\qquad\qquad\qquad\qquad\qquad\qquad\|$

5-dimensional <u>quantum</u> gravity

$\downarrow$

classical limit

$(A_\mu, \varphi^i, i=1,\cdots,6, \chi^A_\alpha, A=1,\cdots,4)$

- Maximally supersymmetric theory in $d=4$ (4 susys)
- β-function g_{YM} is zero
  g_{YM}: <u>true</u> dimensionless param
- conformally invariant
  $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$
  $x^\mu \to x'^\mu(x)$ s.t. $g_{\mu\nu}(x') = \underbrace{\Lambda(x)}_{\text{overall scaling}} g_{\mu\nu}(x)$

$SO(2,d)\Bigg\{$ conformal group:
- <u>Poincare</u>: $P^\mu, M^{\mu\nu}$
- scaling: D
- special conformal: K^μ

For $g_{\mu\nu}=\eta_{\mu\nu}$, such transforms are:

$x'^\mu = x^\mu + a^\mu$

$x'^\mu = \Lambda^\mu_\nu x^\mu$

$x'^\mu = \lambda x^\mu$

$x'^\mu = \dfrac{x^\mu - b^\mu x^2}{1-2b\cdot x + b^2 x^2}$ "special conformal"

$s = \underbrace{I}_{\text{inversion}} \cdot \underbrace{T(b)}_{\text{translation}} \cdot \underbrace{I}_{\text{inversion}}$

(super) <u>Conformal Field Theory</u>

| III

The full bosonic symmetries are

$$SO(2,4) \times \underset{\substack{\uparrow \\ \text{rotate } \varphi^i \text{ (and } \chi^A_\alpha)}}{SO(6)}$$

So with SUSY included, the full (super) group of symmetries is

$$PSU(2,2|4)$$

- Remarks on CFTs

basic objects: <u>local operators</u> with definite scaling dimensions

$$O(x) \to O'(x') = \lambda^{\Delta} O(\lambda x)$$

Δ: dimension

Hilbert space: fall into reps of $SO(2,d)$

typical observables: <u>correlation functions</u> of local operators

conformal symmetry determines the 2 and 3 point correlation functions up to constants:

$$\langle O_1(x) O_2(y) \rangle = \frac{C}{|x-y|^{2\Delta_1}} \delta_{\Delta_1, \Delta_2}$$

$$\langle O_1(x_1) O_2(x_2) O_3(x_3) \rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3} |x_{23}|^{\Delta_2+\Delta_3-\Delta_1} |x_{31}|^{\Delta_3+\Delta_1-\Delta_2}}$$

$x_{ij} = x_i - x_j$

Remarks:

(1) Isometries of $AdS_5 \times S^5$ form a subgroup of general coordinate transformations, which are local symmetries on gravity side

(2) Isometry: This is the subgroup of coordinate transformations that leave the asymptotic form of the metric invariant "large gauge transformations"

(by analogy: $A_\mu \to A_\mu + \partial_\mu \Lambda(x)$
$\Lambda(x) \to 0$ as $|x| \to \infty$ usually
but if $\Lambda(x) \to$ const. as $|x| \to \infty$
it is "large")

- Match parameters:

gravity:		$\mathcal{N}=4$:
$4\pi g_s$	$=$	g_{YM}^2
$R^4/(\alpha')^2$	$=$	$\lambda \equiv g_{YM}^2 N$
G_5/R^3	$=$	$\pi/2N^2$

classical gravity:

$G_5/R^3 \to 0 \quad \Rightarrow N \to \infty \quad \Leftrightarrow$ strong coupling and large N limit

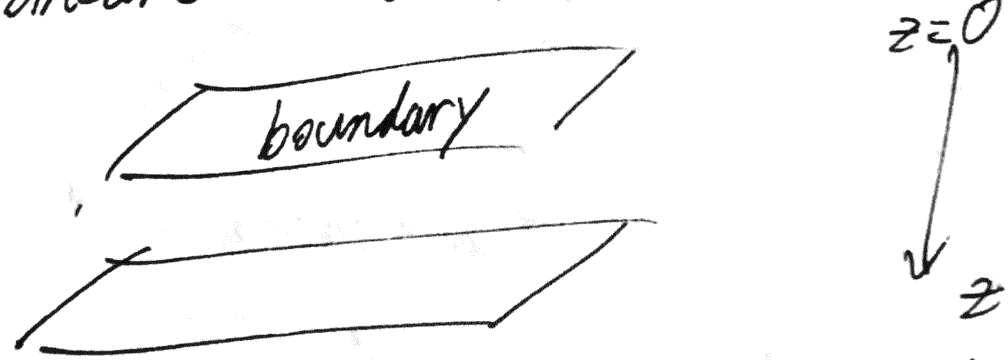
$(\alpha')^2/R^4 \to 0 \quad \Rightarrow \lambda \to \infty$

$1/N^2 \leftrightarrow$ QG corrections

$1/\sqrt{\lambda} \leftrightarrow \frac{\alpha'}{R^2}$ string corrections

- An example of an equivalence between matrices and strings
- can also be considered as an example of the holographic principle

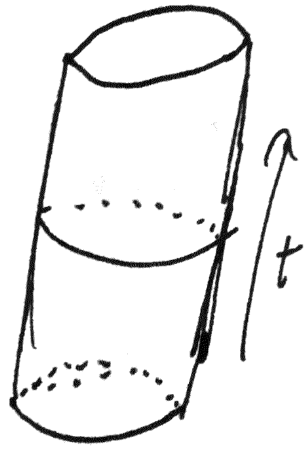
In Poincare coordinates:



$$ds^2 = \frac{R^2}{z^2}(-dt^2 + d\vec{x}^2 + dz^2) \quad z \in (0, \infty)$$

boundary of AdS_5 = $Mink_4$

Holographic perspective $\Rightarrow$ prediction:



boundary of global AdS_5 $= S^3 \times \mathbb{R}$

quantum gravity in global AdS_5
$\parallel$
$\mathcal{N}=4$ SYM on $S^3 \times \mathbb{R}$